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NONSTANDARD HULLS OF BANACH SPACES

BY

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ABSTRACT

The main theme of this paper is the relationship between a Banach space E and its nonstandard hulls $\hat{E}$ (including ultrapowers of E). Emphasis is placed on the ways in which the general structure of $\hat{E}$ is determined by the approximate shape and arrangement of the finite dimensional subspaces of E.

Introduction

The nonstandard hulls of Banach spaces, introduced by Luxemburg [16], are proving to be useful in certain parts of Banach space theory. Also they arise very naturally at many places within nonstandard analysis. The success of any approach which is based on their use depends on having a good understanding of the relationship between a Banach space E and its nonstandard hulls $\hat{E}$. Moreover, the nonstandard hull construction for Banach spaces is just one of many similar constructions in nonstandard analysis. It seems likely that any techniques developed for understanding the relationship between E and $\hat{E}$ will also be useful in these other settings.

Various aspects of this problem have been treated in a series of papers by the author and L. C. Moore, Jr. [2], [5], [6], [9], [10], [11]. Also, similar questions have been considered in the case where the nonstandard hull $\hat{E}$ is actually an ultrapower of E, in papers by Krivine, Dacunha-Castelle and Stern [3], [12], [13], [20], [21], [22], [23]. One important aspect of this program, stated in a very general way, has been to identify pairs of Banach space properties $\mathcal{P}$, $\hat{\mathcal{P}}$ such that for any Banach space E and any of its nonstandard hulls $\hat{E}$:

(A) *E has property $\mathcal{P}$ if and only if $\hat{E}$ has property $\hat{\mathcal{P}}$.*

In practice it has proved to be difficult to identify such pairs of properties or to decide which properties $\mathcal{P}$ or $\hat{\mathcal{P}}$ could be members of such a pair. What is

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needed is a systematic way to analyze the whole question. One fact is obvious, however, and it provides the starting point for such a systematic approach: if $\mathcal{P}$ and $\hat{\mathcal{P}}$ are as in (A), then $\mathcal{P}$ must satisfy

(B) *if E has property $\mathcal{P}$ and E, F have isometric nonstandard hulls, then F has property $\mathcal{P}$ also*

for any Banach spaces E, F.

In part it was this observation which led the author to characterize [6] those pairs of Banach spaces which have isometric nonstandard hulls (constructed using possibly different nonstandard extensions.) One consequence of the main result in [6] is that if $\hat{E}$ is any nonstandard hull of E, then there is an extension $^*\mathcal{M}$ with respect to which E and $\hat{E}$ have isometric nonstandard hulls themselves. It follows from this that if $\mathcal{P}$ is any property which satisfies (B), then for any Banach space E and any of its nonstandard hulls $\hat{E}$

(C) *E has property $\mathcal{P}$ if and only if $\hat{E}$ has property $\mathcal{P}$.*

In particular, this means that satisfying (B) is a necessary and sufficient condition for $\mathcal{P}$ to have a companion property $\hat{\mathcal{P}}$ such that the pair $\mathcal{P}$, $\hat{\mathcal{P}}$ satisfies (A). Moreover, in that case the properties $\mathcal{P}$ and $\hat{\mathcal{P}}$ are equivalent on the class of nonstandard hulls (although not necessarily in general.)

A number of important Banach space properties have been shown to satisfy condition (C) in the papers mentioned above. For example this is true of these properties: super-reflexivity, B-convexity, being an L_p-space ($1 \leqq p < \infty$), being a $\mathcal{L}_p$-space ($1 \leqq p \leqq \infty$). A common feature of these properties is that they can be expressed in terms of finite dimensional subspaces. Let us say somewhat imprecisely that E and F are *finitely equivalent* if they have the same approximate shape and arrangement of finite dimensional subspaces. Also, say that a Banach space property $\mathcal{P}$ is a *local property* if it satisfies

(D) *if E has property $\mathcal{P}$ and if E, F are finitely equivalent, then F has property $\mathcal{P}$ also*

for any Banach spaces E, F. That is, $\mathcal{P}$ is a local property if it depends only on the approximate shape and arrangement of finite dimensional subspaces.

In Section 1 of this paper a precise definition is given of an equivalence relation $\equiv_A$ between Banach spaces, which gives one way to capture the intuitive concept of "finite equivalence". (See Definition 1.6 and the discussion which precedes it.) Moreover, it is proved here that Banach spaces E, F have isometric nonstandard hulls if and only if $E \equiv_A F$. Therefore, it follows that a Banach space property $\mathcal{P}$ which satisfies (B) *must* be a local property, in this sense, and conversely. On the one hand this gives a useful criterion for telling which Banach space properties can occur as in (A). On the other hand, this suggests that nonstandard analysis provides a natural framework for studying the local properties of Banach spaces. (See also [23], where a similar attitude toward "local properties" may be found.)

The discussion above shows that if $\mathcal{P}$ is a local property of Banach spaces, then (A) is satisfied with $\hat{\mathcal{P}}$ equal to $\mathcal{P}$. However, in the most interesting cases the property $\hat{\mathcal{P}}$ is not a local property, and it is the interplay between $\mathcal{P}$ and $\hat{\mathcal{P}}$ which is of importance. Let us say that $\hat{\mathcal{P}}$ can be *localized* if there is a local property $\mathcal{P}$ such that $\mathcal{P}$, $\hat{\mathcal{P}}$ satisfy (A). It is clear that $\hat{\mathcal{P}}$ can be localized if and only if

(E) *either all nonstandard hulls of E have property $\hat{\mathcal{P}}$ or none of them do*

for every Banach space E. When $\hat{\mathcal{P}}$ satisfies (E) then we can formally *define* a local property $\mathcal{P}$ which "localizes" $\hat{\mathcal{P}}$ (in the sense that $\mathcal{P}$, $\hat{\mathcal{P}}$ satisfy (A)) by the equivalence:

E has property $\mathcal{P}$ if and only if every nonstandard hull of E has property $\hat{\mathcal{P}}$.

This is, however, a definition of $\mathcal{P}$ which is in general likely to be difficult to translate into Banach space theory terms. At present we do not have a good general theory of properties which can be localized. To develop such a theory seems important.

One useful observation [11] is that the nonstandard hulls of E all have the same separable subspaces, up to isometry. Therefore, any property $\hat{\mathcal{P}}$ which depends only on separable subspaces can be localized. For example this is true for the property of reflexivity; here the corresponding local property is super-reflexivity [9]. The papers mentioned above contain many other suggestive examples of this type.

In a sense then, the main underlying theme of this paper and of the study of nonstandard hulls in general, is to come to an understanding of local Banach space properties and of the process of localization, where possible. Our approach to the problem is to adopt a model-theoretic point of view toward Banach spaces. That is, we treat Banach spaces as structures for a first-order language L and consider them as models for the sentences in L. Beyond this, it is necessary to single out a special class of *positive bounded* formulas in L and to define a new relation $\models_A$ of *approximate truth* of such formulas in Banach spaces.

Section 1 of this paper is devoted to the initial development of this point of view, showing how it is relevant to the study of nonstandard hulls. Somewhat surprisingly, this model theory for Banach spaces seems to develop in close parallel to general model theory itself; we plan to pursue this further in a somewhat broader setting in [7]. This is unexpected since the class of Banach spaces is not characterized by any first-order properties; moreover, the class of positive bounded logical formulas which we consider is not closed under negation. Nonetheless, it seems that essentially every aspect of general model theory has a significant parallel in this setting. The aspect of this theory which is developed here may be viewed as parallel to the use of highly saturated models in general model theory.

In Sections 2 and 3 we give detailed consideration to the nonstandard hulls of the L_p-spaces ($1 \leqq p < \infty$) and the $C(X)$ spaces. A complete classification under finite equivalence is given for the L_p-spaces and the structure of their nonstandard hulls is determined in a more or less complete way. A similar analysis is given for the $C(X)$ spaces, where X is totally disconnected. It is also shown that any two infinite dimensional L_p-spaces or $C(X)$ spaces have isomorphic nonstandard hulls. (These are related to some interesting results of Stern [23] for L_p-spaces and $C(X)$ spaces. We are grateful to Stern for making available preprint copies of this and other work.)

Preliminaries

The basic framework of nonstandard analysis is developed in [16], [17] and [18] and we assume familiarity with these details. (See also [24], which gives a comprehensive and up-to-date account of the subject.) Throughout this paper we take $\mathcal{M}$ to be a set-theoretical structure, as in one of these presentations, which contains the set $\mathbf{C}$ of complex numbers, and also the sets $\mathbf{R}$ and $\mathbf{N}$ of real numbers and natural numbers, respectively. We denote by ${}^*\mathcal{M}$ an appropriate nonstandard extension of $\mathcal{M}$; thus ${}^*\mathcal{M}$ satisfies the Transfer Principle, which simply asserts that ${}^*\mathcal{M}$ is an elementary extension of $\mathcal{M}$ in a suitable sense. Also ${}^*\mathcal{M}$ satisfies the non-triviality assumption, that ${}^*\mathbf{N} \neq \mathbf{N}$ (or, equivalently, that ${}^*\mathbf{R}$ contains non-zero infinitesimal numbers).

In addition, we assume throughout this paper that ${}^*\mathcal{M}$ is an $\aleph_1$-saturated extension of $\mathcal{M}$ [16]. This condition can be expressed in either of the following, equivalent ways:

(i) If $\{A_n \mid n \in \mathbf{N}\}$ is a collection of internal sets with the finite intersection property, then $\cap \{A_n \mid n \in \mathbf{N}\} \neq \emptyset$.

(ii) If A is an internal set, then any function from $\mathbf{N}$ into A is the restriction of an internal function from ${}^*\mathbf{N}$ into A.

The nonstandard hull construction in its most general form applies to objects in $^*\mathcal{M}$ which are internal Banach spaces. These are internal vector spaces S, equipped with an internal function ρ from S into the nonstandard scalar field, such that the usual Banach space axioms are satisfied, as interpreted in the context of $^*\mathcal{M}$. Put another way, (S,ρ) is an internal Banach space if there is a set $\mathcal{B}$ of Banach spaces in $\mathcal{M}$ such that (S,ρ) is an element of $^*\mathcal{B}$.

Given an internal Banach space (S,ρ), we say $p \in S$ is *finite* if $\rho(p)$ is a finite scalar; p is *infinitesimal* if $\rho(p)$ is infinitesimal. Denote the vector space of finite elements of S by $\mathrm{fin}(S)$. Then define $\hat{S}$ to be the quotient space of $\mathrm{fin}(S)$ obtained by identifying the infinitesimals to 0; let $\pi: \mathrm{fin}(S) \to \hat{S}$ be the quotient map. Then a norm $\hat{\rho}$ is defined on $\hat{S}$ by setting $\hat{\rho}(x)$ equal to the standard part of $\rho(p)$, where $x = \pi(p)$. Then $(\hat{S},\hat{\rho})$ is a standard Banach space over the standard field of scalars, **R** or **C**. The key role of the $\aleph_1$-saturation assumption is to insure that $(\hat{S},\hat{\rho})$ is complete [16]. A space $(\hat{S},\hat{\rho})$ constructed in this way will be called a *nonstandard hull.*

If (E,ρ) is a Banach space in $\mathcal{M}$, then $(^*E, {^*\rho})$ is an internal Banach space in $^*\mathcal{M}$. The nonstandard hull constructed as above from $(^*E, {^*\rho})$ is usually denoted by $(\hat{E},\hat{\rho})$. It contains E as a canonical subspace and is called a *nonstandard hull of E.*

An important class of nonstandard hulls is the class of Banach space ultraproducts introduced in [3] and studied extensively by Krivine, Dacunha-Castelle and Stern. (See [12, 13] and [20–23].) As we explain below, these are the nonstandard hulls constructed using an extension $^*\mathcal{M}$ which is obtained from an ultrapower of $\mathcal{M}$ (see [16]). Just as in model theory generally, there are some advantages to be gained by dealing with an explicitly constructed object, such as an ultraproduct, rather than simply using highly saturated models whose existence is guaranteed by the Compactness Theorem or some analogous general result. However, even when dealing with ultraproducts of Banach spaces, the flexible framework of nonstandard analysis is useful, especially as it provides a convenient language for the higher type levels. In any case, there is a close connection between nonstandard analysis and the approach of Krivine, Dacunha-Castelle and Stern.

Recall that the Banach space ultraproduct construction is as follows [3]: Let $\{E_i \mid i \in I\}$ be a family of Banach spaces and let $\mathcal{U}$ be an ω-incomplete ultrafilter on I. Consider functions α defined on I such that for each $i \in I$ the value $\alpha(i)$ lies in E_i and such that there is a uniform bound on the norms of the elements $\{\alpha(i) \mid i \in I\}$. Two such functions α_1, α_2 are equivalent if the $\mathcal{U}$-limit of the norms of $\{\alpha_1(i) - \alpha_2(i) \mid i \in I\}$ is 0. The Banach space ultraproduct E is the vector space of all equivalence classes $[\alpha]$ of such functions α. The norm of $[\alpha]$ is defined to be the $\mathcal{U}$-limit of the norms of $\{\alpha(i) \mid i \in I\}$. To see that this is a nonstandard hull, let $\mathcal{M}$ be any structure which contains the family $\{E_i \mid i \in I\}$ and let $^*\mathcal{M}$ be the extension of $\mathcal{M}$ constructed using the ultrafilter $\mathcal{U}$. (See [16].) Then $^*\mathcal{M}$ is an $\aleph_1$-saturated extension of $\mathcal{M}$. Moreover, there exists $p \in {}^*I$ such that $p \in {}^*J$ for every set J in $\mathcal{U}$. We consider the internal Banach space $^*E_p$. By the construction of $^*\mathcal{M}$, we know that $^*E_p$ corresponds exactly to the usual ultraproduct $\mathcal{U}$-prod $\langle E_i \mid i \in I\rangle$. It is routine to show that the equivalence classes $[\alpha]$ in E correspond exactly to the elements of the nonstandard hull constructed from $^*E_p$. In particular, they are isometric as Banach spaces.

For simplicity we consider explicitly only Banach spaces over $\mathbf{R}$. All of our results are valid for complex spaces also, and it is routine to extend the arguments. We use standard Banach space terminology and notation, as used in the recent book [14] for example.

1. Positive bounded formulas

The most useful approach to the problem of analyzing the relationship between a Banach space and its nonstandard hulls is to regard Banach spaces as structures for certain first-order languages and to study their model-theoretical properties. This was done in [5] in order to obtain certain isometries between specific nonstandard hulls, and also in [6] in order to give a characterization of those pairs of Banach spaces which have isometric nonstandard hulls. This point of view will be continued here in order to give a more systematic account of the relation between a Banach space and its nonstandard hulls.

Let L be the first-order language whose nonlogical symbols are: a binary function symbol $+$, two unary predicate symbols P and Q and for each rational number r a unary function symbol f_r. We regard each Banach space E (with norm ρ) as an L-structure by taking $+_E$ to be the vector addition on E, by setting

$$P_E = \{x \mid \rho(x) \leqq 1\}$$

$$Q_E = \{x \mid \rho(x) \geqq 1\}$$

and by taking $(f_r)_E$ to be the operation of scalar multiplication by r, for each rational number r.

If t is a term of L and r is a rational number, then we will often write $r \cdot t$ or rt in place of $f_r(t)$. Similarly if φ is the interpretation of f_r in some L-structure, we will use ra in place of $\varphi(a)$. Usually we will refer to the interpretation of $+$ in an L-structure simply as $+$, and will write $a+b$ for $+(a,b)$.

Let T be the theory of all non-trivial Banach spaces, considered as L-structures. It is clear that for any term t in L with variables $x_1,\cdots,x_n$ we can effectively find rational numbers $r_1,\cdots,r_n$ so that

$$\vdash_T t=(r_1x_1+\cdots+r_nx_n).$$

(Here we should take the term $r_1x_1+\cdots+r_nx_n$ to be grouped in some fixed way. Relative to T the particular grouping does not matter.)

Suppose E is a Banach space with norm ρ and t is a term $r_1x_1+\cdots+r_nx_n$ in L. If $a_1,\cdots,a_n$ are elements of E, then the interpretation of the atomic formulas $P(t)$ and $Q(t)$ in E is evidently given by

$$E\models P(t)[a_1,\cdots,a_n]\Leftrightarrow \rho(\Sigma r_ia_i)\leqq 1$$

$$E\models Q(t)[a_1,\cdots,a_n]\Leftrightarrow \rho(\Sigma r_ia_i)\geqq 1.$$

Also if r is any positive rational number, then the norm inequalities

$$\rho(\Sigma r_ia_i)\leqq r \text{ and } \rho(\Sigma r_ia_i)\geqq r$$

can be expressed using $P(1/r\cdot t)$ and $Q(1/r\cdot t)$ respectively. An atomic formula $t=s$ simply expresses a specific linear dependency among the named elements. Note that in T it is equivalent to an atomic formula $t'=0$.

An arbitrary quantifier-free formula σ can be written as a disjunction $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_n$, where each σ_j is a conjunction of atomic formulas or their negations. In this paper we will be concerned mainly with positive formulas; in that case each σ_j is just a conjunction of atomic formulas. Thus each σ_j in σ expresses a finite number of norm estimates of the forms

$$\rho(\Sigma r_ia_i)\leqq r$$

$$\rho(\Sigma s_ia_i)\geqq s$$

together with a finite number of linear equalities of the form

$$\Sigma t_ia_i=0.$$

(And in each, only specific rational coefficients occur.) Certain special classes of formulas in L will be of importance here. Recall that a formula is *positive* if it can be built up from atomic formulas using only conjunction, disjunction and universal or existential quantifiers. A formula is *positive bounded* if it can be built up from atomic formulas using conjunction, disjunction and the *bounded* quantifiers

$$(\exists x)(Px \wedge \cdots)$$

$$(\forall x)(Px \rightarrow \cdots).$$

The results in [6] are given in terms of positive formulas of L, while here we will make use of positive bounded formulas. Informally the difference is simply that bounded quantifiers are restricted to range over the closed unit ball, while unbounded quantifiers range over the entire space. Since a description of the entire space can be easily obtained from its restriction to the unit ball, it is not immediately obvious what change in expressive power is entailed by using the bounded quantifiers. We will show below that each positive formula is equivalent in T to a positive bounded formula, when the free variables are restricted to range over the closed unit ball. Thus positive bounded formulas are, if anything, more expressive than positive formulas. Our other reasons for using positive bounded formulas are twofold. First, we are able to give a very detailed analysis of the truth of positive bounded formulas in nonstandard hulls, an analysis which does not seem to be directly available for positive formulas. Second, our treatment of positive bounded formulas extends directly to situations where Banach spaces with operators, orderings, etc. are considered. In those contexts the restriction to use bounded quantifiers seems to be essential.

The language L is actually an expansion of the language used in [6]; for technical convenience we have included in L symbols for scalar multiplications by rational numbers. Let L_0 be the language, contained in L, whose nonlogical symbols are $+$, P and Q; let T_0 be the restriction of T to L_0. Evidently T_0 is equal to the theory of nontrivial Banach spaces considered as L_0-structures. In [6] the language and theory used were just L_0 and T_0.

As was observed in [6], if $\mathcal{A}$ is a model of T_0 then there is a unique way to regard $(|\mathcal{A}|, +)$ as a vector space over the rational numbers. Moreover, the scalar multiplication operations are first-order definable in $\mathcal{A}$ in a way which does not depend on $\mathcal{A}$. It follows by an easy argument that T is an extension by definitions of T_0 (see [19, section 4.6]). That is, the theory obtained from T_0 by adding the axioms giving first-order definitions for the operations f_r is exactly the theory T.

We are justified then in regarding the theories T and T_0 as being the same. For example, if σ is any formula of L then there is an effectively found formula σ_0 in L_0 such that

$$\vdash_T \sigma \leftrightarrow \sigma_0.$$

Moreover, each model of T_0 has a unique expansion which is a model of T.

Now let $\mathscr{A}$ be an arbitrary model of T, so $|\mathscr{A}|$ is a vector space over the rational numbers under the addition $+_{\mathscr{A}}$ and the scalar multiplication operations $(f_r)_{\mathscr{A}}$. Recall that we will write the values of these operations as $a+b$ and ra, respectively. As in [6], an element a of $|\mathscr{A}|$ is said to be *finite* if there is an integer $n>0$ such that $(1/n)a \in P_{\mathscr{A}}$. The set of finite elements of $|\mathscr{A}|$ is the domain of a substructure of $\mathscr{A}$ which will be denoted by $\mathrm{fin}(\mathscr{A})$.

LEMMA 1.1. *If $\mathscr{A}$ is a model of T, then* $\mathrm{fin}(\mathscr{A})$ *is an elementary substructure of $\mathscr{A}$.*

PROOF. This follows from [6, lemma 1] and the fact that T is an extension by definitions of T_0.

THEOREM 1.2. *For each positive formula σ with free variables $x_1, \cdots, x_n$, there is a positive bounded formula τ with the same free variables such that*

$$\vdash_T Px_1 \wedge \cdots \wedge Px_n \rightarrow (\sigma \leftrightarrow \tau).$$

PROOF. We may assume that σ is in prenex form. We argue by induction on the number of quantifiers in σ. The result is obvious for quantifier-free formulas. We will give details only for the part of the induction step involving existential quantifiers. Universal quantifiers are treated similarly.

Now assume that σ is $(\exists x)\sigma_1$ and that the result holds for all positive formulas with fewer quantifiers than σ. For each integer $m>1$ let σ_m be the result of replacing all free occurrences of x in σ_1 by the term $m \cdot x$. By the induction assumption, for each $m>1$ there is a positive bounded formula τ_m with free variables $x, x_1, \cdots, x_n$ such that

$$\vdash_T Px \wedge Px_1 \wedge \cdots \wedge Px_n \rightarrow (\sigma_m \leftrightarrow \tau_m).$$

Let τ'_m be the result of replacing each free occurrence of x in τ_m by $(1/m)x$. Then

$$\vdash_T P\left(\frac{1}{m}x\right) \wedge Px_1 \wedge \cdots \wedge Px_n \rightarrow (\sigma_1 \leftrightarrow \tau'_m)$$

for each $m \geqq 1$. Note that this implies that if $1 \leqq k \leqq m$, then

$$\vdash_T P\left(\frac{1}{m}x\right) \wedge Px_1 \wedge \cdots \wedge Px_n \rightarrow (\tau'_k \leftrightarrow \tau'_m).$$

Let α_m be the formula

$$Px_1 \wedge \cdots \wedge Px_n \rightarrow \left[(\exists x)\sigma_1 \leftrightarrow (\exists x)\left(P\left(\frac{1}{m}x\right) \wedge \tau'_m\right)\right].$$

It will suffice to show that α_m is a theorem of T for some $m \geqq 1$, since

$$(\exists x)\left(P\left(\frac{1}{m}\right) \wedge \tau'_m\right)$$

is equivalent in T to a positive bounded formula. We note that if $1 \leqq k \leqq m$, then

$$\vdash_T \alpha_k \rightarrow \alpha_m.$$

Therefore, if no α_m is provable in T there is a model $\mathscr{A}$ of T and $a_1, \cdots, a_n \in |\mathscr{A}|$ so that

$$\mathscr{A} \models \neg \alpha_m [a_1, \cdots, a_n]$$

for every $m \geqq 1$. It follows that $a_1, \cdots, a_n$ are in $P_{\mathscr{A}}$. Moreover, there must exist $b \in |\mathscr{A}|$ such that

$$\mathscr{A} \models \sigma_1[b, a_1, \cdots, a_n]$$

since

$$\vdash_T (\exists x)\left(P\left(\frac{1}{m}x\right) \wedge \tau'_m\right) \rightarrow (\exists x)\sigma_1$$

for all $m \geqq 1$. By Lemma 1.1 we may assume that b is finite in $\mathscr{A}$, so there exists $m \geqq 1$ with $(1/m)b \in P_{\mathscr{A}}$. But this implies

$$\mathscr{A} \models (\exists x)\left(P\left(\frac{1}{m}x\right) \wedge \tau'_m\right) [b, a_1, \cdots, a_n]$$

and hence

$$\mathscr{A} \models \alpha_m [a_1, \cdots, a_n]$$

which is a contradiction.

COROLLARY 1.3. *For each positive sentence σ there is a positive bounded sentence τ such that $\vdash_T \sigma \leftrightarrow \tau$.*

Although the proof given here does not show it, the formula τ in Theorem 1.2 can be found effectively from σ. This can be shown by giving a recursively axiomatized theory T' contained in T for which the argument above is still valid. (We will not describe T' here, but it is not hard to extract the needed axioms from the discussion above and the proof of [6, lemma 1].) It would be more satisfactory to have an explicit way of constructing τ from σ, as we do not now have.

It should also be noted that where Banach spaces with extra relations and operations are considered, the analogues of Lemma 1.1 and Theorem 1.2 may fail. In those contexts one must restrict attention to positive bounded formulas from the start. With some mild conditions on the extra relations and operations, the entire theory we develop here will carry through, as will be shown in [7].

Now let σ be an arbitrary positive bounded formula of L. We wish to construct a sequence of other positive bounded formulas σ_m^+ (for $m \geqq 1$) which "approximate" σ in a certain sense. Having fixed an integer $m \geqq 1$, we obtain σ_m^+ from σ by making the following replacements for those atomic formulas which are not part of a bounded quantifier:

(i) replace $t = s$ by $P(m \cdot (t - s))$;

(ii) replace $P(t)$ by $P((1 - (1/m)) \cdot t)$;

(iii) replace $Q(t)$ by $Q((1 + (1/m)) \cdot t)$.

That is, when σ is an atomic formula we define σ_m^+ using (i), (ii) or (iii). For more complex formulas we use the identities

$$(\sigma \wedge \tau)_m^+ = \sigma_m^+ \wedge \tau_m^+$$

$$(\sigma \vee \tau)_m^+ = \sigma_m^+ \vee \tau_m^+$$

$$(\exists x)(Px \wedge \sigma)_m^+ = (\exists x)(Px \wedge \sigma_m^+)$$

$$(\forall x)(Px \rightarrow \sigma)_m^+ = (\forall x)(Px \rightarrow \sigma_m^+)$$

and proceed inductively.

In each of the cases (i), (ii), (iii) above, the given atomic formula is replaced by an atomic formula which is provably weaker in T. It follows that $\sigma \rightarrow \sigma_m^+$ is a theorem of T for each $m \geqq 1$. Also, for similar reasons

$$\vdash_T \sigma_{m+1}^+ \rightarrow \sigma_m^+$$

holds for each m. (Here it is essential that only $\wedge$, $\vee$ and quantifiers are used in building up formulas.) Informally we think of the formulas σ_m^+ as representing approximations to σ, with the degree of accuracy in the approximation improving as m increases. We are led to introduce a relation $\models_A$ of "approximate truth" as follows.

Definition 1.3. *Let σ be a positive bounded formula of L with n free variables. For each L-structure $\mathscr{A}$ and each $a_1, \cdots, a_n \in |\mathscr{A}|$ we say*

$$\mathscr{A} \models_A \sigma[a_1, \cdots, a_n] \text{ if and only if } \mathscr{A} \models \sigma_m^+[a_1, \cdots, a_n]$$

for every $m \geqq 1$.

LEMMA 1.4. *Let σ be any positive bounded formula in L with n free variables.*

(i) *For each $m \geqq 1$*

$$\vdash_T \sigma \rightarrow \sigma_m^+ \text{ and } \vdash_T \sigma_{m+1}^+ \rightarrow \sigma_m^+$$

(ii) *Let $\mathscr{A}$ be any model of T. For each $m \geqq 1$ and $a_1, \cdots, a_n \in |\mathscr{A}|$,*

$$\mathscr{A} \vDash \sigma[a_1, \cdots, a_n] \text{ implies } \mathscr{A} \vDash_A \sigma[a_1, \cdots, a_n].$$

PROOF. Part (i) was remarked above; Part (ii) is an immediate consequence of (i) and the definition of $\vDash_A$.

It seems useful to include here a few informal remarks on the interpretation of $\vDash_A$ in a Banach space E with norm ρ. First suppose σ is an atomic formula $P(t)$. As discussed above we may consider t to be of the form $\Sigma r_i x_i$ with $r_1, \cdots, r_n$ rational numbers. In that case, σ_m^+ corresponds to the estimate:

$$\rho(\Sigma r_i x_i) \leqq 1 + \frac{1}{(m-1)}.$$

Similarly, if σ is $Q(t)$ then σ_m^+ corresponds to $\rho(\Sigma r_i x_i) \geqq 1 - 1/(m+1)$. If σ is $t = 0$ then σ_m^+ corresponds to

$$\rho(\Sigma r_i x_i) \leqq \frac{1}{m}.$$

In each case, when σ is atomic, the formula σ_m^+ expresses a condition which is a genuine approximation of the condition expressed by σ. More generally this is true for all positive, quantifier-free formulas.

LEMMA 1.5. *Let σ be a positive, quantifier-free formula in L with n variables. For each Banach space E and each $a_1, \cdots, a_n \in E$*

$$E \vDash_A \sigma(a_1, \cdots, a_n] \Leftrightarrow E \vDash \sigma[a_1, \cdots, a_n].$$

PROOF. The discussion above makes it clear that this is true when σ is an atomic formula. It is equally obvious that the class of formulas σ for which this equivalence holds is closed under $\wedge$ and $\vee$, which completes the proof.

In general, each positive bounded formula σ is logically equivalent to one of the form

$$Q_1 x_1 \cdots Q_n x_n \tau$$

where $Q_1, \cdots, Q_n$ represent bounded quantifiers and τ is a positive, quantifier-free formula. By Lemma 1.5 and the discussion preceding it, each τ_m^+ expresses an approximate version of the system of norm estimates and equalities expressed by τ. Moreover, if $\mathscr{A}$ is any L-structure and $a_1, \cdots, a_k \in |\mathscr{A}|$ then by definition:

$$\mathcal{A} \models_A Q_1 x_1 \cdots Q_n x_n \tau [a_1, \cdots, a_k]$$

if and only if for each $m \geqq 1$

$$\mathcal{A} \models Q_1 x_1 \cdots Q_n x_n (\tau_m^+) [a_1, \cdots, a_k].$$

This discussion shows that the properties expressible in terms of $\models_A$ and positive bounded formulas are those which concern the approximate shape and arrangement of finite sets of elements. In each single expression there is a finite upper limit on the number of elements discussed (the number of variables in the formula) and only finitely many norm estimates are allowed. This suggests the following definition.

DEFINITION 1.6. *Let $\mathcal{A}$, $\mathcal{B}$ be L-structures. We say $\mathcal{A}$ and $\mathcal{B}$ are finitely equivalent and write $\mathcal{A} \equiv_A \mathcal{B}$ if, for each positive bounded sentence σ*

$$\mathcal{A} \models_A \sigma \Leftrightarrow \mathcal{B} \models_A \sigma.$$

The phrase "finitely equivalent" is chosen to suggest the relation of *finite representability* which is of importance in Banach space theory. Recall that F is finitely representable in E if for each finite dimensional subspace F_0 of F and each $\varepsilon > 0$ there is a finite dimensional subspace E_0 of E and a linear isomorphism T of F_0 onto E_0 such that T and T^{-1} have norm $\leqq 1 + \varepsilon$. That is, the finite dimensional subspaces of F can be embedded into E "almost isometrically". The following theorem, whose proof we omit, gives the connection between this concept and $\models_A$. (See also [11, theor. 2.3].)

We say that a positive bounded formula is *existential* if it consists of a prenex of existential bounded quantifiers followed by a quantifier-free positive formula.

THEOREM 1.7. *Let F and E be Banach spaces. Then F is finitely represented in E if and only if for every existential, positive bounded sentence σ*

$$F \models_A \sigma \text{ implies } E \models_A \sigma.$$

We now turn to the meaning of $\models_A$ in nonstandard hulls. Suppose that $\mathcal{M}$ is fixed and that $^*\mathcal{M}$ is some $\aleph_1$-saturated extension of $\mathcal{M}$. Let S be any internal Banach space in $^*\mathcal{M}$, with internal norm $\rho : S \to {}^*\mathbf{R}$. Just as done for Banach spaces themselves, we regard S as an L-structure: $+_S$ is the given internal addition on S,

$$P_S = \{p \mid p \in S \text{ and } \rho(p) \leqq 1\}$$

$$Q_S = \{p \mid p \in S \text{ and } \rho(p) \geqq 1\}$$

and for each standard rational number r, $(f_r)_S$ is the given internal operation of scalar multiplication by r on S. Note that the Transfer Principle insures that S is a model of T. Hence the substructure of finite elements of S, $\mathrm{fin}(S)$ is an elementary substructure of S. Recall that π denotes the canonical quotient homomorphism of $\mathrm{fin}(S)$ onto $\hat{S}$.

LEMMA 1.8. *Let S be an internal Banach space in $^*\mathcal{M}$ and let $p_1, \cdots, p_n \in S$ be finite. Let σ be any positive bounded formula in L with n free variables.*

(i) $S \models \sigma[p_1, \cdots, p_n]$ *implies* $\hat{S} \models \sigma[\pi(p_1), \cdots, \pi(p_n)]$.

(ii) *For each* $m \geqq 1$

$$\hat{S} \models \sigma^+_{m+1}[\pi(p_1), \cdots, \pi(p_n)] \textit{ implies } S \models \sigma^+_m[p_1, \cdots, p_n].$$

PROOF.

(i) The mapping π is a homomorphism of $\mathrm{fin}(S)$ onto $\hat{S}$ as L-structures; moreover, π maps P_S onto $P_{\hat{S}}$. The general fact that truth of positive formulas is preserved under homomorphisms shows that

$$\mathrm{fin}(S) \models \sigma[p_1, \cdots, p_n] \text{ implies } \hat{S} \models \sigma[\pi(p_1), \cdots, \pi(p_n)].$$

But $P_S = P_{\mathrm{fin}(S)}$, so that the truth of any formula whose quantifiers are relativized to P is the same in $\mathrm{fin}(S)$ as in S. This proves (i).

(ii) Let ρ be the internal norm on S. It is clear from the definition of $\hat{\rho}$ on $\hat{S}$ that if $0 < r < s$ are rational numbers and $p \in S$ is finite, then

$$\hat{\rho}(\pi p) \leqq r \text{ implies } \rho(p) < s$$

and

$$\hat{\rho}(\pi p) \geqq s \text{ implies } \rho(p) > r.$$

From this and the fact that π is a homomorphism it is immediate that (ii) holds when σ is an atomic formula. A straightforward inductive argument shows that (ii) holds in general.

THEOREM 1.9. *Let S be an internal Banach space in $^*\mathcal{M}$ and let $p_1, \cdots, p_n$ be finite. For each positive bounded formula σ in L with n free variables, the following conditions are equivalent*:

(i) $\hat{S} \models \sigma[\pi(p_1), \cdots, \pi(p_n)]$;

(ii) $\hat{S} \models_A \sigma[\pi(p_1), \cdots, \pi(p_n)]$;

(iii) $S \models_A \sigma[p_1, \cdots, p_n]$.

PROOF. The equivalence between (ii) and (iii) follows directly from Lemma 1.8. Also, Lemma 1.4 (ii) shows that (i) implies (ii). Thus it remains only to show

(iii) implies (i); we do this by induction on the logical complexity of σ. We note first that if $p \in S$ is finite, then

$$\rho(p) \leqq m/(m-1) \text{ for all } m \text{ implies } \hat{\rho}(\pi(p)) \leqq 1$$

and

$$\rho(p) \geqq m/(m+1) \text{ for all } m \text{ implies } \hat{\rho}(\pi(p)) \geqq 1$$

and

$$\rho(p) \leqq 1/m \text{ for all } m \text{ implies } \pi(p) = 0.$$

This shows that (iii) implies (i) for all atomic formulas σ. The induction steps concerning disjunction, conjunction and universal bounded quantifiers are trivial. It remains to consider σ of the form $(\exists x)(Px \wedge \tau)$ where the stated result holds for τ. Condition (iii) for σ asserts that for each $m \geqq 1$

$$S \models (\exists x)(Px \wedge \tau_m^+)[p_1, \cdots, p_n].$$

Consider for each $m \geqq 1$ the internal set

$$A_m = \{p \in S \mid S \models \tau_m^+[p, p_1, \cdots, p_n] \text{ and } \rho(p) \leqq 1\}.$$

By assumption each A_m is nonempty and by Lemma 1.4(i)

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots.$$

Since ${}^*\mathcal{M}$ is assumed to be $\aleph_1$-saturated, it follows that there exists $p \in S$ satisfying $\rho(p) \leqq 1$ and

$$S \models \tau_m^+[p, p_1, \cdots, p_n]$$

for all $m \geqq 1$. The induction assumption yields

$$\hat{S} \models \tau[\pi p, \pi p_1, \cdots, \pi p_n]$$

and hence

$$\hat{S} \models \sigma[\pi p_1, \cdots, \pi p_n].$$

This completes the proof.

Evidently Theorem 1.9 gives a complete characterization of truth in the nonstandard hull $\hat{S}$ in terms of truth in the internal object S, for positive bounded formulas. Moreover, it illustrates how central the approximate truth relation $\models_A$ is in studying nonstandard hulls. In particular, in any nonstandard hull, the relations $\models$ and $\models_A$ coincide, for positive bounded formulas.

COROLLARY 1.10. *Let E be a Banach space and $\hat{E}$ one of the nonstandard hulls of E. For each positive bounded formula σ with n free variables and each $a_1, \cdots, a_n \in E$, the following conditions are equivalent:*

(i) $\hat{E} \models \sigma[a_1, \cdots, a_n]$;

(ii) $\hat{E} \models_A \sigma[a_1, \cdots, a_n]$;

(iii) $E \models_A \sigma[a_1, \cdots, a_n]$.

PROOF. We apply Theorem 1.9, where S is $^*E$. This shows that (i) and (ii) are equivalent and that they are equivalent to:

$$^*E \models_A \sigma[^*a_1, \cdots, ^*a_n].$$

The Transfer Principle implies that this is equivalent to (iii).

COROLLARY 1.11. *Each Banach space is finitely equivalent to each of its nonstandard hulls.*

COROLLARY 1.12. *Let E be a Banach space and $a_1, \cdots, a_n \in E$. For each positive bounded formula σ with n free variables*

$$E \models_A \sigma[a_1, \cdots, a_n] \Leftrightarrow E \models_A \sigma^+_m[a_1, \cdots, a_n] \text{ for every } m \geqq 1.$$

PROOF. Let $\hat{E}$ be any nonstandard hull of E. By Corollary 1.10, the statement to be proved is equivalent to

$$\hat{E} \models_A \sigma[a_1, \cdots, a_n] \Leftrightarrow \hat{E} \models \sigma^+_m[a_1, \cdots, a_n] \text{ for all } m \geqq 1.$$

But this is true by definition.

THEOREM 1.13. *For each pair E, F of Banach spaces, the following conditions are equivalent:*

(i) $E \equiv_A F$;

(ii) *There is a Banach space H such that each positive bounded sentence true in E or in F is true in H;*

(iii) *E and F have isometric nonstandard hulls;*

(iv) *E and F have isometric Banach space ultrapowers.*

PROOF. It was shown in [6] that (iii) and (iv) are equivalent. Also it was shown there that (iii) and (iv) are equivalent to a condition which, by Theorem 1.2, is weaker than (ii). To prove (i) implies (ii) we need only take H to be a nonstandard hull of E. If σ is a positive bounded sentence and σ is true in E or in F, then $E \models_A \sigma$. Hence by Corollary 1.10 it follows that $H \models \sigma$, as desired.

Finally, that (iii) implies (i) follows from Corollary 1.11 and the fact that isometric spaces are finitely equivalent.

COROLLARY 1.14. *Let E be a Banach space and κ an infinite cardinal number. For each set $S \subseteq E$ of cardinality $\leqq \kappa$ there is a closed subspace F of E such that $S \subseteq F$, F has density character $\leqq \kappa$ and $F \equiv_A E$.*

PROOF. Using the Downward Löwenheim-Skolem Theorem [19], obtain an elementary substructure $\mathscr{A}$ of E which contains S and has cardinality $\leqq \kappa$. Let F be the closure of $\mathscr{A}$ in E. Evidentally $S \subseteq F$ and F has a dense subset of cardinality $\leqq \kappa$.

Choose $\mathscr{M}$ to contain E and let $^*\mathscr{M}$ be an extension of $\mathscr{M}$ which has the $\aleph_0$-isomorphism property. It is shown in [5] that, under these hypotheses, the nonstandard hulls of F and E are isometric. Therefore, $F \equiv_A E$ by Theorem 1.13.

Actually, in some situations it is useful to know that certain specific nonstandard hulls are isometric. For this it is convenient to assume that $^*\mathscr{M}$ has one of the isomorphism properties introduced in [5].

THEOREM 1.15. *Let $^*\mathscr{M}$ be an extension of $\mathscr{M}$ which has the $\aleph_0$-isomorphism property. If S_1, S_2 are internal Banach spaces in $^*\mathscr{M}$ and $\hat{S}_1 \equiv_A \hat{S}_2$, then $\hat{S}_1$ and $\hat{S}_2$ are isometric.*

In particular, if E, F are finitely equivalent Banach spaces in $\mathscr{M}$ and $\hat{E}$, $\hat{F}$ are the nonstandard hulls constructed using $^*\mathscr{M}$, then $\hat{E}$ is isometric to $\hat{F}$.

PROOF. Let S_1, S_2 be internal Banach spaces in $^*\mathscr{M}$ and suppose $\hat{S}_1 \equiv_A \hat{S}_2$. By Theorem 1.9 we know that a positive bounded sentence is true in $\hat{S}_1$ if and only if it is true in $\hat{S}_2$. By Theorem 1.2 this is true also of positive sentences of L. As shown in the proof of [6, theor. 2] there is a Banach space E in $\mathscr{M}$ such that exactly the same sentences are true in $\hat{S}_1$ as in E.

Now let σ be a positive sentence of L which is true in S_1 or in S_2. By Theorem 1.2 and Theorem 1.9, σ is true in $\hat{S}_1$ or $\hat{S}_2$; therefore σ is true in E. By the Transfer Principle σ is true in $^*E$. In [6] it was shown that in this situation, when $^*\mathscr{M}$ has the $\aleph_0$-isomorphism property, there are homomorphisms onto $^*E$ from S_1 and from S_2, and these homomorphisms induce isometries between $\hat{S}_1$, $\hat{S}_2$ and $\hat{E}$. (See for example the proof of [6, prop. 2].) This proves the first part of this theorem. The second part follows immediately, using Corollary 1.11 to see that if $E \equiv_A F$, then $\hat{E} \equiv_A \hat{F}$.

One apparent disadvantage of working with the set of positive bounded formulas is that it is not closed under negation. For example, there exist Banach spaces E and F such that every positive bounded sentence true in E is true in F, but *not* conversely. That is, the set of positive bounded sentences true in given space need not be maximal in the way that such a "logical theory" should be. The next result shows that this is not a problem when we consider $\models_A$ in place of $\models$.

COROLLARY 1.16 *If E and F are Banach spaces such that for any positive bounded sentence*

$$E \models \sigma \text{ implies } F \models_A \sigma,$$

then E and F are finitely equivalent.

PROOF. Suppose σ is a positive bounded sentence and $E \models \sigma$. Then for each $m \geqq 1$, $E \models \sigma_m^+$ and hence $F \models_A \sigma_m^+$ by assumption. By Corollary 1.12 we see that $F \models_A \sigma$.

Now let $\hat{E}$, $\hat{F}$ be any nonstandard hulls of E, F respectively. By Corollary 1.10, any positive bounded sentence true in $\hat{E}$ is true in $\hat{F}$. Therefore $\hat{E}$ and $\hat{F}$ are finitely equivalent, by Theorem 1.9. In turn this implies that E and F are finitely equivalent, by Corollary 1.11.

Now let $\{E_i \mid i \in I\}$ be an indexed family of Banach spaces and $\mathcal{U}$ an ω-incomplete ultrafilter on I; let E be the Banach space ultraproduct of the family $\{E_i \mid i \in I\}$, constructed using $\mathcal{U}$. As explained above, if S denotes the ordinary ultraproduct of this family, as L-structures, then S may be regarded as an internal Banach space (relative to an appropriate $\mathcal{M}$ and $^*\mathcal{M}$) and E is then simply the nonstandard hull $\hat{S}$. Thus we may use Theorem 1.9 to analyze the truth of positive bounded formulas in E.

Recall that elements in E may be regarded as equivalence classes of functions α defined on I such that $\alpha(i) \in E_i$ for all $i \in I$ and such that there is a uniform bound on the norms of the elements $\alpha(i)$. Let $[\alpha]$ denote the equivalence class of such an α, as an element of E. Let $\alpha/\mathcal{U}$ denote the equivalence class of α as an element of the ordinary ultraproduct S.

THEOREM 1.17. *Let E be the Banach space ultraproduct of the family $\{E_i \mid i \in I\}$ of Banach spaces, relative to the ω-incomplete ultrafilter $\mathcal{U}$ on I. For each positive bounded formula in L with n free variables and each $[\alpha_1], \cdots, [\alpha_n]$ in E, the following are equivalent:*

(i) $E \models \sigma[[\alpha_1], \cdots, [\alpha_n]]$;

(ii) $E \models_A \sigma[[\alpha_1], \cdots, [\alpha_n]]$;

(iii) *for each* $m \geqq 1$

$$\{i \mid E_i \models \sigma_m^+[\alpha_1(i), \cdots, \alpha_n(i)]\} \in \mathcal{U}.$$

PROOF. The discussion above and Theorem 1.9 show that (i) and (ii) are equivalent to each other and to the condition: for each $m \geqq 1$

$$S \models \sigma_m^+[\alpha_1/\mathcal{U}, \cdots, \alpha_n/\mathcal{U}],$$

where S is the ordinary ultraproduct of $\{E_i \mid i \in I\}$. This condition is equivalent to (iii) by the fundamental property of ultraproducts.

COROLLARY 1.18. (*Compactness Theorem*). *Let Σ be a set of positive bounded sentences such that for each $m \geqq 1$ and each finite subset Σ_0 of Σ, there is a Banach space E_0 such that*

$$E_0 \models \sigma_m^+ \text{ for all } \sigma \in \Sigma_0.$$

Then there are Banach spaces (of arbitrarily large cardinality) such that

$$E \models \sigma \text{ for all } \sigma \in \Sigma.$$

PROOF. Let I be the set of all pairs (Σ_0, m) where Σ_0 is a finite subset of Σ and $m \geqq 1$. For each $(\Sigma_0, m) = i$ in I let E_i be a Banach space chosen so that $E_i \models \sigma_m^+$ for all $\sigma \in \Sigma_0$. There is an ultrafilter $\mathcal{U}$ on I which contains each of the sets

$$\{(\Sigma_0', m') \mid \Sigma_0 \subseteq \Sigma_0' \text{ and } m \leqq m'\}$$

where (Σ_0, m) is a fixed element of I. Let E be the Banach space ultraproduct of $\{E_i \mid i \in I\}$ constructed using $\mathcal{U}$. Theorem 1.17 implies that $E \models \sigma$ for all $\sigma \in \Sigma$. By Corollary 1.10, the nonstandard hulls of E provide arbitrarily large spaces with the desired property.

As noted in the preliminaries, the construction of Banach space ultraproducts is included in a more general construction. Namely, take an indexed family $\{E_i \mid i \in I\}$ of Banach spaces, choose $p \in {}^*I$ and take E to be the nonstandard hull of the internal Banach space ${}^*E_p$. It is an easy matter to prove the analogue of Theorem 1.17 for this construction. Having done so, it follows that E is finitely equivalent to the Banach space ultraproduct of the family $\{E_i \mid i \in I\}$ constructed using the ultrafilter $\mathcal{U}$ on I which is determined by p. (That is, $A \subseteq I$ is in $\mathcal{U}$ iff $p \in {}^*A$.) For this and other reasons we have used ultraproducts here in place of the more general construction.

DEFINITION 1.19. *Let $\mathcal{C}$ be a class of Banach spaces.*

(i) *$\mathcal{C}$ is a **local class** if whenever E is in $\mathcal{C}$ and F is finitely equivalent to E, then F is in $\mathcal{C}$.*

(ii) $\mathscr{C}$ *is* ***compact*** *if for each set* Σ *of positive bounded sentences the following holds: if for each* $m \geqq 1$ *and each finite subset* Σ_0 *of* Σ *there is* $E_0 \in \mathscr{C}$ *so that*

$$E_0 \models \sigma_m^+ \text{ for all } \sigma \in \Sigma_0$$

then there exists $E \in \mathscr{C}$ *such that*

$$E \models_A \sigma \text{ for all } \sigma \in \Sigma.$$

According to Definition 1.19, a class or property of Banach spaces is *local* if it depends only on the approximate shape and arrangement of the finite dimensional subspaces, *to the extent that is expressible using positive bounded sentences.* One may view this definition as being somewhat restrictive, in that for each particular first-order formula there is a specific upper bound on the number of elements mentioned (equivalently, on the dimension of subspaces considered). Thus, according to this definition, the following condition does not seem to define a local class:

For each x_1, x_2 in E, of norm $\leqq 1$, there exists a finite set of elements $x_3, \cdots, x_n$, also of norm $\leqq 1$, such that for some choice of signs

$$\| \pm x_1 \pm x_2 \pm x_3 \pm \cdots \pm x_n \| \geqq \frac{2}{3}.$$

If a specific n is chosen, then this condition can be expressed in the form

$$(\forall x_1 \forall x_2 \exists x_3 \cdots \exists x_n \sigma)_2^+.$$

If not, then there does not seem to be a way to express this condition in terms of any set of positive bounded sentences.

This simply points out that there are perhaps many ways to make precise the concept of local class or property of Banach spaces. The one discussed here is the most appropriate in connection with nonstandard hulls and Banach space ultraproducts. We hope to return to this point in the future. In particular, if $\mathscr{C}$ is a class of Banach spaces which is closed under Banach space ultraproducts (more generally, if $\mathscr{C}$ is compact), then it seems that $\mathscr{C}$ must be local in the sense of Definition 1.19 if it is local in any reasonable sense. In particular, the distinctions discussed above do not matter when $\mathscr{C}$ is compact.

We note that a local class of Banach spaces is closed under isometry. Moreover, the results above show that $\mathscr{C}$ is a local class if and only if it is closed under isometry and satisfies the condition:

$$E \text{ is in } \mathscr{C} \Leftrightarrow \hat{E} \text{ is in } \mathscr{C}$$

for each Banch space E and each nonstandard hull $\hat{E}$ of E. (It is equivalent here to restrict $\hat{E}$ to range over the ultrapowers of E, since we know that finitely equivalent Banach spaces have isometric ultrapowers, by Theorem 1.13.) Also, Theorem 1.17 (and the proof of Corollary 1.18) show that any class which is closed under Banach space ultraproducts is a compact class.

The result which follows gives a characterization of compact, local classes in more algebraic terms and also as the analogue in this setting of elementary classes in model theory.

Given a set Σ of positive bounded sentences, define

$$\mathrm{Mod}_A(\Sigma) = \{E \mid E \text{ is a Banach space and } E \models_A \sigma \text{ for all } \sigma \in \Sigma\}.$$

Also, if E is a Banach space, define

$$\mathrm{Th}_A(E) = \{\sigma \mid \sigma \text{ is a positive bounded sentence and } E \models_A \sigma\}.$$

THEOREM 1.20. *For a class $\mathscr{C}$ of Banach spaces, the following conditions are equivalent*:

(i) *$\mathscr{C}$ is a compact, local class*;

(ii) *$\mathscr{C}$ is closed under Banach space ultraproducts, is closed under isometry and satisfies*

$$E \in \mathscr{C} \Leftrightarrow \hat{E} \in \mathscr{C}$$

for each Banach space E and each Banach space ultrapower $\hat{E}$ of E;

(iii) *For some set Σ of positive bounded sentences, $\mathscr{C} = \mathrm{Mod}_A(\Sigma)$.*

PROOF. The discussion above shows that (ii) implies (i). Theorem 1.17, Corollary 1.11 and the fact that every Banach space ultrapower of E is a nonstandard hull of E, show that (iii) implies (ii).

It remains to show that (i) implies (iii). Let $\mathscr{C}$ be a compact, local class and let

$$\Sigma = \{\sigma \mid \sigma \text{ is a positive bounded sentence and } E \models_A \sigma \text{ for all } E \in \mathscr{C}\}.$$

It suffices to show that $\mathrm{Mod}_A(\Sigma) \subseteq \mathscr{C}$.

In order to prove this we need to introduce some technical machinery which is also useful in other contexts. Let σ be a positive bounded formula and $m \geqq 1$. We introduce a new "approximant" of σ, to be denoted by $\sigma_m^{\sim}$, which is defined by induction on the logical complexity of σ:

$P(t)^-_m$	is	$\neg Q((1-(1/m))\cdot t)$
$Q(t)^-_m$	is	$\neg P((1+(1/m))\cdot t)$
$(t=s)^-_m$	is	$\neg Q(m\cdot(t-s))$
$(\sigma\wedge\tau)^-_m$	is	$\sigma^-_m\wedge\tau^-_m$
$(\sigma\vee\tau)^-_m$	is	$\sigma^-_m\vee\tau^-_m$
$(\exists x)(Px\wedge\sigma)^-_m$	is	$(\exists x)(Px\wedge\sigma^-_m)$
$(\forall x)(Px\rightarrow\sigma)^-_m$	is	$(\forall x)(Px\rightarrow\sigma^-_m)$.

LEMMA 1.21. *For each positive bounded formula σ and each $m\geqq 1$:*
(i) $\neg\sigma^-_m$ *is logically equivalent to a positive bounded formula;*
(ii) $\vdash_T\sigma^-_m\rightarrow\sigma^+_m$ *and* $\vdash_T\sigma^+_{m+1}\rightarrow\sigma^-_m$.

PROOF OF LEMMA 1.21.
(i) This is immediate from the definition by an inductive argument.
(ii) The sentences $(\forall x)(\neg Qx\rightarrow Px)$ and $(\forall x)(\neg Px\rightarrow Qx)$ are easily seen to be theorems of T, as are $(\forall x)(P(rx)\rightarrow\neg Q(sx))$ and $(\forall x)(Q(sx)\rightarrow\neg P(rx))$, when $0<s<r$ are rational numbers. This shows that (ii) holds whenever σ is an atomic formula. An inductive argument shows that (ii) holds in general.

Now we complete the proof of Theorem 1.20. Let E be a Banach space in $\mathrm{Mod}_A(\Sigma)$. Fix $\sigma\in\mathrm{Th}_A(E)$ and let $m\geqq 1$. There must be an F in $\mathscr{C}$ such that σ^+_m holds in F. For otherwise, by Lemma 1.21, $F\models\neg\sigma^-_m$ for every $F\in\mathscr{C}$; in that case $\neg\sigma^-_m$ is logically equivalent to a sentence in Σ, so that $E\models\neg\sigma^-_m$. But $E\models\sigma^+_{m+1}$, which is a contradiction by Lemma 1.21.

Let Σ_0 be a finite subset of $\mathrm{Th}_A(E)$ and $m\geqq 1$. Applying the argument above to the conjunction of Σ_0, we see that there exists F in $\mathscr{C}$ such that

$$F\models\sigma^+_m \text{ for all } \sigma\in\Sigma_0.$$

Since $\mathscr{C}$ is a compact class, there exists F in $\mathscr{C}$ such that $\mathrm{Th}_A(E)\subseteq\mathrm{Th}_A(F)$. But Corollary 1.16 yields that E and F must be finitely equivalent. Since $\mathscr{C}$ is a local class, this shows that E is in $\mathscr{C}$, which completes the proof of Theorem 1.20.

In dealing with specific classes of Banach spaces, condition (ii) in Theorem 1.20 is often straightforward to check. It has been shown to hold for many important classes, such as the classes of L_p-spaces $(1\leqq p<\infty)$, L_1-preduals; more generally for the $\mathscr{L}_{p,\lambda+}$-spaces $(1\leqq p\leqq\infty, \lambda\geqq 1)$ and many others. (See [10, for example and also [3], [12], [13] and [21].) For many of these classes $\mathscr{C}$ an explicit set of positive bounded sentences Σ satisfying $\mathscr{C} = \mathrm{Mod}_A(\Sigma)$ can be extracted from known results, especially those of Krivine, Dacunha-Castelle and Stern.

If $\mathscr{C}_0$ is an arbitrary class of Banach spaces it is easy to see that there must be a smallest compact, local class $\mathscr{C}$ which contains $\mathscr{C}_0$. If we let

$$\Sigma = \{\sigma \mid \sigma \text{ is a positive bounded sentence and } E \models_A \sigma \text{ for all } E \in \mathscr{C}_0\}$$

then $\mathscr{C}$ must equal $\mathrm{Mod}_A(\Sigma)$, by Theorem 1.20. The following result gives a different kind of characterization of $\mathscr{C}$.

THEOREM 1.22. *Let $\mathscr{C}_0$ be a class of Banach spaces and let $\mathscr{C}$ be the smallest compact, local class of Banach spaces which contains $\mathscr{C}_0$. Then a Banach space E is in $\mathscr{C}$ if and only if some Banach space ultrapower of E is isometric to a Banach space ultraproduct of members of $\mathscr{C}_0$.*

PROOF. Since $\mathscr{C}$ is a compact, local class it satisfies condition (ii) of Theorem 1.20. Therefore it contains E whenever a Banach space ultrapower of E is isometric to a Banach space ultraproduct of members of $\mathscr{C}_0 \subseteq \mathscr{C}$.

Conversely, suppose E is in $\mathscr{C}$. That is, $E \models_A \sigma$ whenever σ is a positive bounded sentence such that $F \models_A \sigma$ for every $F \in \mathscr{C}_0$. Arguing as in the proof of Theorem 1.20, there exists a Banach space ultraproduct of members of $\mathscr{C}_0$, say F, which is finitely equivalent to E. By Theorem 1.13 these are Banach space ultrapowers $\hat{E}$, $\hat{F}$ of E, F respectively such that $\hat{E}$ and $\hat{F}$ are isometric. Now $\hat{F}$ is a Banach space ultrapower of a Banach space ultraproduct of members of $\mathscr{C}_0$. The proof is completed by showing that such an $\hat{F}$ must actually be a Banach space ultraproduct of members of $\mathscr{C}_0$. This is straightforward to prove and we omit the details. (See [16, p. 54] for example.)

We close this section by posing two questions which are suggested by the results presented here and which seem to involve aspects of both Banach space theory and model theory in important and interesting ways.

(1) Let $\mathscr{C}_0$ be the class of Banach spaces which are isometric to some Banach space $C(X)$, of all continuous, real-valued functions on a compact Hausdorff space X. It can be shown that $\mathscr{C}_0$ is closed under Banach space ultraproducts (see Section 3.). Let $\mathscr{C}$ be the smallest compact, local class which contains $\mathscr{C}_0$. By Theorem 1.22, a Banach space E is in $\mathscr{C}$ if and only if some ultrapower of E is in $\mathscr{C}_0$. Equivalently, E is in $\mathscr{C}$ if and only if there is a compact, Hausdorff space X such that $E \equiv_A C(X)$.

One open question is whether $\mathscr{C}_0$ is a proper subset of $\mathscr{C}$. If so, as seems likely, then $\mathscr{C}$ is a natural class of Banach spaces to investigate. For example, by [10, theor. 2.2] each member of $\mathscr{C}$ is an $\mathscr{L}_{\infty,1+}$-space, that is, a space whose dual space is isometric to an L_1-space. Yet, as will be shown in [7], not every such space is in $\mathscr{C}$.

We remark that the same problem seems interesting when $\mathscr{C}_0$ is the class of Banach spaces which can be given Banach lattice structure.

(2) Let $\mathscr{C}_0$ be the class of all finite dimensional Banach spaces, and $\mathscr{C}$ the smallest compact, local class of Banach spaces which contains $\mathscr{C}_0$. From Theorem 1.22 we have:

(i) $E \in \mathscr{C}$ if and only if some Banach space ultrapower of E is isometric to a Banach space ultraproduct of finite dimensional spaces. Or, in the language of [11], E is in $\mathscr{C}$ if and only if some nonstandard hull of E is isometric to a hyperfinite dimensional nonstandard hull. In [5] it is shown that the sequence space l_∞ satisfies this condition, and in Section 2 below the same will be proved for l_p $(1 \leqq p < \infty)$. Therefore all the spaces l_p are in $\mathscr{C}$.

Another condition for membership in E is the following:

(ii) $E \in \mathscr{C}$ if and only if for each positive bounded sentence σ and each $m \geqq 1$ there is a finite dimensional space in which σ_m^+ is true.

The problem we pose is to decide which spaces are in $\mathscr{C}$. It seems possible that every Banach space is in $\mathscr{C}$. The simplest spaces for which membership in $\mathscr{C}$ is not settled are the space c_0 of sequences converging to 0 and the space L_1 of Lebesgue integrable functions on [0, 1].

Note that if E is in $\mathscr{C}$, then the question whether E has some given local property can be reduced directly to properties of finite dimensional spaces. Therefore this problem is related to the question whether local properties of Banach spaces can be reduced to facts about finite dimensional spaces.

Section 2. L_p-spaces $(1 \leqq p < \infty)$

Using the ideas of the previous section and some facts from analysis, it is possible to give a complete classification of the L_p-spaces $(1 \leqq p < \infty)$ under $\equiv_A$ and to determine the structure of the nonstandard hulls of L_p-spaces. Before doing this, we summarize a few facts about L_p-spaces and give some notation. As usual we write $L_p(\mu)$ for the Banach space of pth power absolutely integrable (real-valued) functions on some measure space (S, Σ, μ). If μ is Lebesgue measure on [0, 1] we write L_p in place of $L_p(\mu)$. If μ is a purely atomic measure, with $\mu(\{s\}) = 1$ for all $s \in S$, then we write $l_p(S)$ for $L_p(\mu)$. In such a case, when $S = \mathbf{N}$ we just write l_p for $l_p(S)$; if $S = \{1, 2, \cdots, n\}$, then we just write $l_p(n)$. In general we write the norm of $x \in L_p(\mu)$ as $\|x\|$.

On each L_p-space there is a partial ordering $\leqq$ induced by the pointwise ordering of functions, under which the space becomes a Banach lattice. Elements x, y of $L_p(\mu)$ are *disjoint* if $\inf(|x|,|y|)=0$. If x, y are disjoint, $\geqq 0$ and $u = x + y$, then they are called *components* of u. An *atom* of $L_p(\mu)$ is an element $u \geqq 0$ such that $\|u\| = 1$ and the only components of u are 0 and u itself. The closed linear span of the set of atoms is the *purely atomic* part of $L_p(\mu)$. This closed subspace of $L_p(\mu)$ is isometric to $l_p(S)$, where S is the set of atoms. An element x of $L_p(\mu)$ is *purely nonatomic* if $|x|$ has no purely atomic component (except 0). These elements form a closed subspace of $L_p(\mu)$ which is isometric to a space $L_p(\bar{\mu})$. Moreover, each u in $L_p(\mu)$ can be written uniquely as $u = x + y$, where x is purely atomic, y is purely nonatomic and x, y are disjoint. Thus $L_p(\mu)$ is canonically isometric to the direct sum of $l_p(S)$ and $L_p(\bar{\mu})$; moreover, this is an l_p-sum in the sense that

$$\|u\|^p = \|x\|^p + \|y\|^p$$

for $u = x + y$ as above.

Elements x, y of $L_p(\mu)$ are disjoint if and only if

$$\|x + y\|^p = \|x - y\|^p = \|x\|^p + \|y\|^p.$$

Therefore the disjointness of elements is preserved under isometries between L_p-spaces (as Banach spaces), as is the cardinality of the set of atoms. (See [14], for example, for a complete discussion of the L_p-spaces.)

Now let $L_p(\mu)$ be a fixed L_p-space and let $\hat{L}_p(\mu)$ denote one of its nonstandard hulls. As above, let π denote the quotient mapping from the finite part of ${}^*L_p(\mu)$ onto $\hat{L}_p(\mu)$. It was shown in [10] that $\hat{L}_p(\mu)$ is an L_p-space.

LEMMA 2.1. *Let S be the set of atoms of $L_p(\mu)$. Then $\pi({}^*S)$ is the set of atoms of $\hat{L}_p(\mu)$.*

In particular, the purely atomic part of $\hat{L}_p(\mu)$ is isometric to $l_p({}^*S)$.

PROOF. By the Transfer Principle, the elements of ${}^*S$ have norm 1 and are positive in ${}^*L_p(\mu)$. Moreover, since distinct atoms are disjoint, we have

$$\|x - y\|^p = \|x\|^p + \|y\|^p = 2$$

whenever x, y are distinct elements of ${}^*S$. Therefore π maps ${}^*S$ bijectively onto a set of positive elements of $\hat{L}_p(\mu)$ which have norm 1.

Suppose $x \in {}^*S$ and $\pi(x) = u + v$ in $\hat{L}_p(\mu)$, with u, v disjoint and $\geqq 0$. We can choose $u', v' \in {}^*L_p(\mu)$ which are positive, disjoint and satisfy $\pi(u') = u$ and $\pi(v') = v$. Thus $u' + v'$ is infinitely close to x in ${}^*L_p(\mu)$. By splitting u' and v' into purely atomic and nonatomic parts in ${}^*L_p(\mu)$, we see that x must be infinitely close to u' or to v'. That is, either $\pi(x) = u$ or $\pi(x) = v$. This shows that each $\pi(x)$ is an atom in $\hat{L}_p(\mu)$, when $x \in {}^*S$.

If $x \in {}^*L_p(\mu)$ has norm 1, is positive and is not infinitely close to any element of ${}^*S$, then we can find positive u', v' which are disjoint and not infinitesimal in ${}^*L_p(\mu)$ such that $x = u' + v'$. Thus $\pi(x) = \pi(u') + \pi(v')$ and so $\pi(x)$ is not an atom in $\hat{L}_p(\mu)$. This completes the proof.

It was shown in [10] that for a Banach space E with nonstandard hull $\hat{E}$:

$$E \text{ is an } L_p\text{-space if and only if } \hat{E} \text{ is an } L_p\text{-space.}$$

Therefore, by Theorem 1.13, the L_p-spaces are finitely equivalent only to other L_p-spaces. Moreover, by this and Corollary 1.14 each L_p-space is finitely equivalent to a separable L_p-space; the structure of these spaces is well understood.

THEOREM 2.2. *If $L_p(\mu_1)$ and $L_p(\mu_2)$ are infinite dimensional, then $L_p(\mu_1) \equiv_A L_p(\mu_2)$ if and only if one of the following conditions holds:*

(1) *$L_p(\mu_1)$ and $L_p(\mu_2)$ have no atoms;*

(2) *$L_p(\mu_1)$ and $L_p(\mu_2)$ have infinitely many atoms;*

(3) *$L_p(\mu_1)$ and $L_p(\mu_2)$ have the same (finite) number of atoms.*

PROOF. Lemma 2.1 and the preceding discussion makes it clear that one of the conditions must hold if $L_p(\mu_1)$ and $L_p(\mu_2)$ have isometric nonstandard hulls. Thus we have one direction of this result, by Theorem 1.13.

To prove the converse, we may assume that $L_p(\mu_1)$ and $L_p(\mu_2)$ are both separable. In that case, if (1) or (3) holds then $L_p(\mu_1)$ is isometric to $L_p(\mu_2)$ and we are done. If (2) holds, then each $L_p(\mu_i)$ has purely atomic part isometric to l_p. Moreover, if $L_p(\mu_i)$ has a nontrivial purely non-atomic part, then that part is isometric to L_p. It thus remains only to show that l_p is finitely equivalent to the l_p-sum of l_p and L_p.

It is evident that the nonstandard hull $\hat{l}_p$ has nontrivial purely atomic part. Indeed, any finite element of ${}^*l_p$ which takes only infinitesimal values (as a function from ${}^*\mathbf{N}$ into ${}^*\mathbf{R}$) determines a purely nonatomic element of $\hat{l}_p$. Thus we may find (by Corollary 1.14) a separable subspace E of $\hat{l}_p$ such that E contains l_p and also some purely nonatomic elements of $\hat{l}_p$ and satisfies $E \equiv_A \hat{l}_p$. But then E is a separable L_p-space finitely equivalent to l_p and E must be isometric to the l_p-sum of l_p and L_p. This completes the proof.

COROLLARY 2.3. *Any two infinite dimensional L_p-spaces have isomorphic nonstandard hulls.*

PROOF. By Theorem 2.2, any infinite dimensional L_p-space is finitely equivalent to a separable L_p-space which has nontrivial purely nonatomic part. Any such L_p-space is isomorphic to L_p [14]. Thus, by Theorem 1.13, the given L_p-space has nonstandard hulls which are isomorphic to nonstandard hulls of L_p.

In the remainder of this section we examine more closely the structure of the nonstandard hulls $\hat{l}_p$ and $\hat{L}_p$, as L_p-spaces. The previous results show that if ${}^*\mathcal{M}$ is chosen properly (for example if it has the $\aleph_0$-isomorphism property) then the nonstandard hulls of L_p-spaces are isometric to $\hat{l}_p$ or $\hat{L}_p$ on one of the l_p-sums $l_p(n) \oplus \hat{L}_p$. Indeed, under these hypotheses, if S is any internal Banach space and $\hat{S}$ is an L_p-space, then it must be isometric to one of these spaces, by Theorem 1.15.

In order to obtain a simple description of $\hat{l}_p$ and $\hat{L}_p$ it is convenient to make the following hypothesis about ${}^*\mathcal{M}$:

(#) There is a cardinal number κ such that each infinite, internal subset of ${}^*\mathbf{N}$ has (external) cardinality κ.

This condition holds, for example when ${}^*\mathcal{M}$ is obtained via the ultrapower construction using a free ultrafilter on a countable set. In this case $\kappa = 2^{\aleph_0}$. The condition also holds whenever ${}^*\mathcal{M}$ has the $\aleph_0$-isomorphism property. Indeed, in this case all infinite, internal sets have the same (external) cardinality [5].

From now on in this section we suppose that ${}^*\mathcal{M}$ is an extension of $\mathcal{M}$ which satisfies (#) and that $\mathcal{M}$ contains l_p and L_p. We first note that $\hat{l}_p$ and $\hat{L}_p$ have density character κ and cardinality κ. Indeed, let E be any separable, infinite dimensional Banach space in $\mathcal{M}$. We may choose S to be a countable, dense subset of E and choose T to be a countably infinite subset of E whose elements all have norm 1 and are distance $\geqq 1/2$ apart. Then each element of ${}^*E$ is infinitely close to an element of ${}^*S$. Also, the elements of ${}^*T$ determine distinct elements of $\hat{E}$ which all have norm 1 and are distance $\geqq 1/2$ apart. Hence

$$\kappa = \operatorname{card}({}^*T) \leqq \operatorname{density}(\hat{E}) \leqq \operatorname{card}(\hat{E}) \leqq \operatorname{card}({}^*S) = \kappa.$$

Thus $\operatorname{card}(\hat{E}) = \operatorname{density}(\hat{E}) = \kappa$. Also note that the purely atomic part of $\hat{l}_p$ is isometric to $l_p(\Gamma)$, where Γ is a set of cardinality κ (by Lemma 2.1.). The structure of $\hat{l}_p$ and $\hat{L}_p$ is completely determined in the following result. In it we use this notation: $[0,1]^\kappa$ denotes the measure algebra obtained from the product of κ copies of Lebesgue measure; the l_p-sum of an infinite family $\{E_\alpha \mid \alpha \in I\}$ consists of all sequences $\{a_\alpha \mid \alpha \in I\}$ such that $a_\alpha \in E_\alpha$ for all $\alpha \in I$ and $\Sigma_{\alpha \in I} \| a_\alpha \|^p < \infty$. The norm of such a sequence is $(\Sigma \| a_\alpha \|^p)^{1/p}$.

THEOREM 2.4.

1) *$\hat{L}_p$ is isometric to the l_p-sum of κ copies of $L_p([0,1]^\kappa)$.*

2) *The purely nonatomic part of $\hat{l}_p$ is isometric to $\hat{L}_p$.*

PROOF. In each part of the proof we use the well known method for analyzing the structure of an abstract L_p-space, as described in [14, chap. 5] for example. This entails first taking a maximal set S of pairwise disjoint, positive elements of norm 1. (In (2) the elements of S should be purely nonatomic). Each $u \in S$ yields a finite measure μ_u, with μ_u-measurable sets corresponding to components of u. Then the abstract L_p-space being analyzed is isometric to the l_p-sum of the spaces $L_p(\mu_u)$, indexed over $u \in S$. It thus suffices in each case to show:

(a) there is a family S of pairwise disjoint, purely nonatomic, positive elements of norm 1 which has cardinality κ, and

(b) each space $L_p(\mu_u)$ is isometric to $L_p([0,1]^\kappa)$. Moreover, to prove (b) it suffices, by Maharam's Theorem [14], to show:

(b′) for each positive, purely nonatomic element u, there is a set A of components of u and $\delta > 0$ such that $\| x - y \| \geqq \delta$ if $x, y \in A$ are distinct, and $\text{card}(A) = \kappa$.

We will prove (a) and (b′) for the cases (1) and (2) separately, as the details are somewhat different.

(1) In L_p there is a countably infinite set S of pairwise disjoint, positive elements of norm 1. Then $^*S$ determines a set of κ pairwise disjoint, positive elements of $\hat{L}_p$, each of norm 1. This proves (a). To prove (b′) let $u = \pi(x)$ be any positive element of $\hat{L}_p$; we may assume x has norm 1 in $^*L_p$. By using the Transfer Principle and properties of L_p, there is an internal set A of components of x in $^*L_p$ such that if $a, b \in A$ are distinct, then $\| a - b \| \geqq \delta$ where $\delta = (1/2)^{1/p}$. Moreover, A can be taken to be (internally) of the same cardinality as $^*\mathbf{N}$. Then the set $\{\pi(a) \mid a \in A\}$ is a set of κ components of $\pi(x) = u$, with distinct elements being distance $\geqq \delta$ apart. This proves (b′) for $\hat{L}_p$ and completes the proof of (1).

(2) As outlined above, we show that the purely nonatomic part of $\hat{l}_p$ is also isometric to the l_p-sum of κ copies of $L_p([0,1]^\kappa)$. It is an easy matter to produce a set of κ pairwise disjoint, purely nonatomic, positive elements of $\hat{l}_p$. (Note that an element of $^*l_p$ yields a purely nonatomic element of $\hat{l}_p$ if, as a function from $^*\mathbf{N}$ into $^*\mathbf{R}$, its values are all infinitesimal.) For example, let ω be a fixed infinite integer and let I be any internal subset of $^*\mathbf{N}$ which has ω elements. Define x in $^*l_p$ by

$$x(n) = \begin{cases} 0 \text{ if } n \in {}^*\mathbf{N} \sim I \\ (1/\omega)^{1/p} \text{ if } n \in I \end{cases}.$$

Then $\pi(x)$ in $\hat{l}_p$ is a purely nonatomic, positive element of norm 1. Clearly we can produce a family of pairwise disjoint sets I, as above, which is indexed over $^*\mathbf{N}$. Thus we get the desired positive elements of $^*\mathbf{N}$. Now we prove (b′) for the purely nonatomic elements of $\hat{l}_p$. Let x be any element of $^*l_p$ which is positive, of norm 1 and determines a purely nonatomic element $\pi(x)$ of $\hat{l}_p$; that is, for each $n \in {}^*\mathbf{N}$, $x(n)$ is infinitesimal. If k is a standard integer $\geqq 1$, then we can partition $^*\mathbf{N}$ into k sets $A_1, \cdots, A_k$ such that for each $1 \leqq j \leqq k$, the sum

$$\sum_{n \in A_j} |x(n)|^p$$

is infinitely close to $1/k$. It follows that there exist an infinite integer ω, an infinitesimal $\eta > 0$ and an internally indexed partition $\{A_j \mid 1 \leqq j \leqq 2^\omega\}$ of $^*\mathbf{N}$ such that for each j

$$\left| \sum_{n \in A_j} |x(n)|^p - \frac{1}{2^\omega} \right| < \frac{\eta}{2^\omega}.$$

Now we may find an internally indexed family $B_1, B_2, \cdots, B_\omega$ of subsets of $\{1, 2, \cdots, 2^\omega\}$ such that each B_j has $2^{\omega-1}$ elements, as does each symmetric difference $B_i \Delta B_j$ if $i \neq j$. For each $1 \leqq j \leqq \omega$ we define y_j on $^*\mathbf{N}$ by

$$y_j(n) = \begin{cases} x(n) \text{ if } n \in A_i \text{ and } i \in B_j, \text{ some } i \\ \\ 0 \text{ otherwise.} \end{cases}$$

It is an easy computation to show that in $^*l_p$, if $i \neq j$, then $\| y_i - y_j \|^p$ is infinitely close to 1/2. Thus $\{\pi(y_j) \mid 1 \leqq j \leqq \omega\}$ is a set of components of $\pi(x)$ whose distance apart is uniformly $(1/2)^{1/p}$. By the assumption (#) this set of components has cardinality κ. This completes the proof.

For each $\omega \in {}^*\mathbf{N}$, let $\hat{l}_p(\omega)$ denote the subspace of $\hat{l}_p$ which is determined by sequences x in $^*l_p$ which satisfy

$$x(n) = 0 \text{ if } \omega \leqq n.$$

Then $\hat{l}_p(\omega)$ is easily seen to be an abstract L_p-space; moreover, if ω is infinite, then an analysis similar to that given above for $\hat{l}_p$ shows that:

(i) the purely atomic part of $\hat{l}_p(\omega)$ is isometric to $l_p(\Gamma)$, where Γ has cardinality κ;

(ii) the purely nonatomic part of $\hat{l}_p(\omega)$ is isometric to the l_p-sum of κ copies of $L_p([0,1]^\kappa)$.
Therefore, $\hat{l}_p$ is isometric to $\hat{l}_p(\omega)$ for any infinite $\omega \in {}^*\mathbf{N}$.

THEOREM 2.5. *For any positive bounded sentence* σ, $l_p \models_A \sigma$ *if and only if*

$$\text{for each } m \geqq 1 \text{ there exists } N = N(m) \text{ such that } l_p(n) \models \sigma_m^+ \text{ for all } n \geqq N.$$

PROOF. Fix σ. By the argument above, $l_p \models_A \sigma$ if and only if $\hat{l}_p(\omega) \models_A \sigma$ for every infinite $\omega \in {}^*\mathbf{N}$. By Theorem 1.9 this holds if and only if for each $m \geqq 1$,

$$^*l_p(\omega) \models \sigma_m^+ \text{ for all infinite } \omega \in {}^*\mathbf{N}.$$

This yields the desired result by a familiar argument in nonstandard analysis.

We will prove the $p = \infty$ version of Theorem 2.5 in Section 3. It would be very interesting to know if there is any corresponding result possible for L_p. This is closely related to the question (2) raised at the end of Section 1, as the proof of Theorem 2.5 suggests.

We close this section by noting that the methods used in proving Theorem 2.4 can also be used to analyze the structure of the measure spaces introduced by P. Loeb in [15]. To take an important case, let A be an infinite but *-finite set and let λ be an internal, positive function from A into ${}^*\mathbf{R}$ such that each value $\lambda(a)$ is infinitesimal and $\Sigma_{a \in A} \lambda(a)$ is infinitely close to 1. For each internal set $B \subset A$, $\mu(B)$ is defined to be the standard part of the sum $\Sigma_{a \in B} \lambda(a)$. Loeb showed [15] that μ extends uniquely to a σ-additive probability measure (also denoted by μ) on the σ-algebra of subsets of A generated by the internal subsets. Arguing as above, it can be shown that the measure algebra of the finite measure μ is homogeneous, and thus by Maharam's Theorem it is isomorphic to the measure algebra of $[0,1]^\kappa$. In particular, for each p the Banach space $L_p(\mu)$ is isometric to $L_p([0,1]^\kappa)$.

Section 3. $C(X)$ spaces

Let $C(X)$ be the Banach space of all continuous, real-valued functions on X, a compact, Hausdorff space. Let $\mathcal{M}$ be a set-theoretical structure which contains X and $C(X)$ and let ${}^*\mathcal{M}$ be an extension of $\mathcal{M}$; as usual, ${}^*\mathcal{M}$ is $\aleph_1$-saturated over $\mathcal{M}$.

We will show first that the nonstandard hull of $C(X)$ is itself a space of continuous functions on a compact Hausdorff space $\hat{X}$, and we will give a useful description of this space. Each element p of ${}^*C(X)$ is an internal function from

$^*X$ to $^*\mathbf{R}$. Since $C(X)$ is given the supremum norm, p is a finite element of $^*C(X)$ exactly when each value of p is finite in $^*\mathbf{R}$. Therefore, we may define $^\circ p: {}^*X \to \mathbf{R}$ by letting $^\circ p(x)$ be the standard part of $p(x)$, for each $x \in {}^*X$. Let the space of all such functions be denoted by $\hat{C}(X)$. It is a routine exercise to prove the following result.

PROPOSITION 3.1. *$\hat{C}(X)$ is a Banach space when equipped with the supremum norm. Moreover, $\hat{C}(X)$ is isometric to the nonstandard hull of $C(X)$, under the function which maps $^\circ p$ in $\hat{C}(X)$ to $\pi(p)$ in the nonstandard hull.*

We also note that for any finite p and q in $^*C(X)$

$$^\circ(p \cdot q) = {}^\circ p \cdot {}^\circ q$$

$$^\circ(\max(p,q)) = \max({}^\circ p, {}^\circ q)$$

$$^\circ(\min(p,q)) = \min({}^\circ p, {}^\circ q).$$

Therefore $\hat{C}(X)$ is closed under pointwise multiplication and under the pointwise lattice operations. Thus the general theory of continuous function spaces [4] shows that there is a compact, Hausdorff space $\hat{X}$ such that $\hat{C}(X)$ is isometric to $C(\hat{X})$ in a multiplication and lattice preserving way. Indeed, we can identify $\hat{X}$ as the completion of $^*X$ under the $\hat{C}(X)$ uniform structure; that is, under the uniform structure given by the pseudometrics

$$d_f(x,y) = |fx - fy|$$

on $^*X$, where f ranges over $\hat{C}(X)$.

Let τ be the given topology on X. We will refer to the elements of $^*\tau$ as the *internal open* subsets of $^*X$. These sets form the base for an important topology on $^*X$ called the Q-topology. We now show that it is this topology which is induced on $^*X$ as a subspace of $\hat{X}$. This is the same as showing that the topology on $^*X$ induced by the $\hat{C}(X)$ uniformity is the Q-topology. Suppose first that $x \in {}^*X$, $f \in \hat{C}(X)$ and $\delta > 0$ is a standard real number. Then $f = {}^\circ p$ for some p in $^*C(X)$. The set $S = \{y \in {}^*X \mid |fy - fx| < \delta\}$ contains $\{y \in {}^*X \mid |py - px| < \delta/2\}$ which is an internal open set. Thus S is a neighborhood of x in the Q-topology. This shows that each f in $\hat{C}(X)$ is continuous relative to the Q-topology. Next, suppose S is any neighborhood of x for the Q-topology. There must be an internal open set $\mathcal{O}$ such that $x \in \mathcal{O} \subseteq S$. By transferring Urysohn's Lemma to $^*\mathcal{M}$, we see that there must be $p \in {}^*C(X)$ which takes its values in $^*[0,1]$, has value 0 at x and has value 1 on $^*X \sim \mathcal{O}$. Letting $f = {}^\circ p$, we have

$$x \in \{y \in {}^*X \mid |fy - fx| < 1/2\} \subseteq \mathcal{O}.$$

Therefore, each open set in the Q-topology is open relative to the $\hat{C}(X)$-uniformity. We have proved:

PROPOSITION 3.2. *There is a compact, Hausdorff space $\hat{X}$ which contains ${}^*X$ as a dense subset and such that:*

(1) *the subspace topology on ${}^*X$ is the Q-topology;*

(2) *each function in $\hat{C}(X)$ extends to a real-valued continuous function on $\hat{X}$;*

(3) *the restrictions of each real-valued continuous function on $\hat{X}$ is in $\hat{C}(X)$.*

Evidently the correspondence between $C(\hat{X})$ and $\hat{C}(X)$ described in Proposition 3.2 is an isometry of Banach spaces and preserves the pointwise multiplication and lattice operations. This result and Proposition 3.1 shows that $C(\hat{X})$ may be canonically regarded as identical to the nonstandard hull of $C(X)$. Consider the statement $C(X) \equiv_A C(Y)$, where X and Y are compact, Hausdorff spaces. By Theorem 1.13, this statement holds if and only if ${}^*\mathcal{M}$ can be chosen so that $C(X)$ and $C(Y)$ have isometric nonstandard hulls. This is the same as saying that $C(\hat{X})$ and $C(\hat{Y})$ are isometric, and this is equivalent to saying that $\hat{X}$ is homeomorphic to $\hat{Y}$.

If X is a compact, Hausdorff space, let $\mathcal{B}(X)$ be the Boolean algebra of subsets of X which are closed and open (clopen). Next we analyze the algebra $\mathcal{B}(\hat{X})$ of clopen subsets of $\hat{X}$.

PROPOSITION 3.3. *A subset of $\hat{X}$ is in $\mathcal{B}(\hat{X})$ if and only if it is the closure of an element of ${}^*\mathcal{B}(X)$.*

PROOF. First let A be an element of ${}^*\mathcal{B}(X)$. That is, both A and ${}^*X \sim A$ are internal open sets. The characteristic function p of A

$$p(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in {}^*X \sim A \end{cases}$$

is in ${}^*C(X)$ and so $f = {}^\circ p$ is in $\hat{C}(X)$. Also let f denote the continuous extension of ${}^\circ p$ to $\hat{X}$. Since ${}^*X$ is dense in $\hat{X}$, f takes on just the values 0 and 1 on $\hat{X}$. Thus $S = \{x \in \hat{X} \mid f(x) = 1\}$ is in $\mathcal{B}(\hat{X})$. Moreover, S is the closure of A in $\hat{X}$.

Next suppose S is in $\mathcal{B}(\hat{X})$. Then the characteristic function f of S is in $C(\hat{X})$. By Proposition 3.2 there is $p \in {}^*C(X)$ so that on ${}^*X$, f equals ${}^\circ p$. Therefore, if $x \in {}^*X$, then $p(x)$ is infinitely close to 0 or to 1. It follows that $A =$

$\{x \in {}^*X \mid f(x) = 0\}$ is also equal to $\{x \in {}^*X \mid p(x) < 1/3\}$ and ${}^*X \sim A = \{x \in {}^*X \mid p(x) > 2/3\}$. Therefore A and ${}^*X \sim A$ are internal open sets, so A is an element of ${}^*\mathcal{B}(X)$. Moreover S must be the closure of A, by the definition of A and the fact that ${}^*X$ is dense in $\hat{X}$. This completes the proof.

COROLLARY 3.4. *Let X and Y be compact, Hausdorff spaces and suppose $C(X) \equiv_A C(Y)$. If X is connected, then so is Y.*

PROOF. If $C(X) \equiv_A C(Y)$, then as discussed above, ${}^*\mathcal{M}$ can be chosen so that $\hat{X}$ and $\hat{Y}$ are homeomorphic. If X is connected, then $\mathcal{B}(X)$ has just two elements. Thus ${}^*\mathcal{B}(X)$ has two elements also. By Proposition 3.3 it follows that $\mathcal{B}(\hat{X})$ has two elements. Hence the same is true of $\mathcal{B}(\hat{Y})$, ${}^*\mathcal{B}(Y)$ and $\mathcal{B}(Y)$. Thus Y is also connected.

It is clear that many similar results can be proved by this kind of argument. One which involves an additional idea is the following:

PROPOSITION 3.5. *Let X and Y be compact, Hausdorff spaces and suppose $C(X) \equiv_A C(Y)$. If X is totally disconnected, then so is Y.*

PROOF. Following the pattern of the previous proof, it suffices to show that for any compact, Hausdorff space X, X is totally disconnected if and only if $\hat{X}$ is totally disconnected.

First suppose $\hat{X}$ is totally disconnected and let $x \neq y$ be in X. Then there is a clopen subset S of $\hat{X}$ so that ${}^*x \in S$ and ${}^*y \notin S$. By Proposition 3.3 there is a set $A \subseteq {}^*X$ such that A and ${}^*X \sim A$ are internal open sets, ${}^*x \in A$ and ${}^*y \notin A$. By transferring the existence of such a set back to $\mathcal{M}$, we see that there is a clopen set $\mathcal{O} \subseteq X$ such that $x \in \mathcal{O}$ and $y \notin \mathcal{O}$. Since x and y were arbitrary and since X is compact, it follows that X is totally disconnected.

Conversely, suppose that X is totally disconnected and x, y are distinct elements of $\hat{X}$. There is a function f in $C(\hat{X})$ such that $f(x) = 0$ and $f(y) = 1$. For some finite element p in ${}^*C(X)$, f equals ${}^\circ p$ on ${}^*X$, by Proposition 3.2. Consider the two disjoint internal closed sets $\{z \in {}^*X \mid p(z) \leqq 1/3\}$ and $\{z \in {}^*X \mid p(z) \geqq 2/3\}$. Since X is totally disconnected (and using the Transfer Principle) there is an $A \subseteq {}^*X$ such that A and ${}^*X \sim A$ are internal open sets and $A \supseteq \{z \in {}^*X \mid p(z) \leqq 1/3\}$ and ${}^*X \sim A \supseteq \{z \in {}^*X \mid p(z) \geqq 2/3\}$. Let S be the closure of A in $\hat{X}$. By Proposition 3.3, S is a clopen set. Moreover, since ${}^*X$ is dense in $\hat{X}$ we see that $f(x) = 0$ implies that x is in S, and $f(y) = 1$ implies that y is not in S. That is, any pair of distinct points in $\hat{X}$ can be separated by a clopen set. Since $\hat{X}$ is compact, this shows that $\hat{X}$ is totally disconnected, completing the proof.

THEOREM 3.6. *Let X, Y be compact, Hausdorff spaces and suppose that X is totally disconnected. Then $C(X) \equiv_A C(Y)$ if and only if*

(1) Y is totally disconnected, and

(2) the Boolean algebras $\mathscr{B}(X)$, $\mathscr{B}(Y)$ are elementarily equivalent.

PROOF. Suppose first $C(X) \equiv_A C(Y)$. By Proposition 3.5, Y must be totally disconnected. By Theorem 1.13, $^*\mathscr{M}$ can be chosen so that the nonstandard hulls of $C(X)$ and $C(Y)$ are isometric. By Proposition 3.1 and 3.2 this implies that $\hat{X}$ and $\hat{Y}$ are homeomorphic. Hence $^*\mathscr{B}(X)$ and $^*\mathscr{B}(Y)$ are isomorphic, by Proposition 3.3. But $\mathscr{B}(X)$ is elementarily equivalent to $^*\mathscr{B}(X)$ and $\mathscr{B}(Y)$ to $^*\mathscr{B}(Y)$; hence $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ are elementarily equivalent to each other.

Conversely, suppose $\mathscr{B}(X)$, $\mathscr{B}(Y)$ are elementarily equivalent. Choose $\mathscr{M}$ to include X, Y, $C(X)$ and $C(Y)$ and let $^*\mathscr{M}$ be an extension of $\mathscr{M}$ which has the $\aleph_0$-isomorphism property [5]. Then $^*\mathscr{B}(X)$ is isomorphic to $^*\mathscr{B}(Y)$. Since X and Y are assumed to be totally disconnected, the same is true of $\hat{X}$ and $\hat{Y}$, by Proposition 3.5. By Proposition 3.3 the algebras $\mathscr{B}(\hat{X})$ and $\mathscr{B}(\hat{Y})$ are isomorphic, which implies that $\hat{X}$ and $\hat{Y}$ are homeomorphic, by the duality theory for Boolean algebras. Therefore, by Propositions 3.1 and 3.2, $C(X)$ and $C(Y)$ have isometric nonstandard hulls. This proves $C(X) \equiv_A C(Y)$ by Theorem 1.13.

COROLLARY 3.7. *Let X be any infinite, compact, Hausdorff space.*

(1) $C(X) \equiv_A l_\infty$ if and only if X is totally disconnected and has a dense subset of isolated points.

(2) Let Δ be the Cantor set. $C(X) \equiv_A C(\Delta)$ if and only if X is totally disconnected and has no isolated points.

PROOF.

1) l_∞ is $C(\beta\mathbf{N})$, where $\beta\mathbf{N}$ is the Stone-Cech compactification of the countable discrete space $\mathbf{N}$. The Boolean algebra $\mathscr{B}(\beta\mathbf{N})$ is the power set of $\mathbf{N}$, an infinite, atomic Boolean algebra. Any two such algebras are elementarily equivalent [1]. Therefore, by Theorem 3.6, $C(X) \equiv_A l_\infty$ if and only if X is totally disconnected and $\mathscr{B}(X)$ is infinite and atomic. This is the same as saying that X is infinite and has a dense set of isolated points.

2) $\mathscr{B}(\Delta)$ is an atomless Boolean algebra, and any two such algebras are elementarily equivalent [1]. A totally disconnected space X has an atomless algebra $\mathscr{B}(X)$ if and only if X has no isolated points.

REMARKS.

(1) A complete classification of the elementary equivalence classes of Boolean algebras was given by Tarski [25]. (See also [1].) There are only $\aleph_0$ such classes.

(2) The results of this section can all be easily extended to include the multiplication on $C(X)$. See [8] for some discussion of full elementary equivalence of $C(X)$ spaces.

Next we complete the analysis of the theory of l_p spaces begun in Theorem 2.5 by considering the case $p=\infty$. For $n \in \mathbf{N}$, $l_\infty(n)$ is the n-dimensional sequence space, under the supremum norm.

THEOREM 3.8. *For any positive bounded sentence σ, $l_\infty \models_A \sigma$ if and only if for each $m \geqq 1$ there exists $N = N(m)$ such that $l_\infty(n) \models \sigma_m^+$ for all $n \geqq N$.*

PROOF. Choose $\mathcal{M}$ to contain l_∞ and let $^*\mathcal{M}$ be an extension of $\mathcal{M}$ which has the $\aleph_0$-isomorphism property. It was shown in [5, p. 730] that in this setting, $\hat{l}_\infty$ is isometric to $\hat{l}_\infty(\omega)$ for each infinite $\omega \in {}^*\mathbf{N}$. Now argue as in the proof of Theorem 2.5.

Next we prove a result, for $C(X)$ spaces, of Löwenheim-Skolem type. (Compare Corollary 1.14 above and also see [8, theor. 9.3].)

THEOREM 3.9. *Let X be a compact, Hausdorff space and κ an infinite cardinal number. For each set $S \subseteq C(X)$ of cardinality $\leqq \kappa$ there is a closed subspace F of $C(X)$ and a compact, Hausdorff space Y such that $S \subseteq F$, F has density character $\leqq \kappa$, $F \equiv_A C(X)$ and F is isometric to $C(Y)$.*

PROOF. We argue as in the proof of Corollary 1.14, but using an expanded first-order language which contains a relation symbol corresponding to the partial ordering $\leqq$ on $C(X)$. Relative to this language, take an elementary substructure $\mathcal{A}$ of $C(X)$ which contains S, contains the constant function 1 and has cardinality $\leqq \kappa$. In particular, $\mathcal{A}$ is necessarily a sublattice of $C(X)$. As before, let F be the closure of $\mathcal{A}$ in $C(X)$. The only new fact to be proved about F is that it is isometric to $C(Y)$ for some compact, Hausdorff space Y. But this follows from the fact that F is a closed sublattice of $C(X)$ which contains the constant function 1 [14].

COROLLARY 3.10. *For each infinite, compact Hausdorff space X there is an uncountable compact metric space Y such that*

$$C(X) \equiv_A C(Y).$$

PROOF. Every Banach space is finitely represented in $C(X)$. Therefore, by [11, theor. 2.3] the nonstandard hull $C(\hat{X})$ of $C(X)$ contains a subspace isometric to $C(0,1)$. Let S be a countable dense subset of this subspace and apply Theorem 3.9 to $C(\hat{X})$, taking $\kappa = \aleph_0$. This yields a compact, Hausdorff space Y such that $C(Y)$ is separable, $C(Y)$ contains a subspace isometric to $C(0,1)$ and

$$C(Y) \equiv_A C(\hat{X}) \equiv_A C(X).$$

But then Y is a metric space, since $C(Y)$ is separable, and Y is uncountable, since $C(Y)$ contains $C(0,1)$. Therefore Y is the desired space.

Milutin's Theorem asserts that if X, Y are uncountable, compact metric spaces, then $C(X)$ is isomorphic to $C(Y)$ [14]. From this and Corollary 3.10 the following result is immediate, using Theorem 1.13. (See also Corollary 2.3, which is the analogous result for L_p-spaces.)

COROLLARY 3.11. *Any two infinite dimensional $C(X)$ spaces have isomorphic nonstandard hulls.*

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